

Electromagnetic and gravitational radiation from massless particles

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We demonstrate that full description of both electromagnetic and gravitational radiation from massless particles lies outside the scope of classical theory. Synchrotron radiation from the hypothetical massless charge in quantum electrodynamics in external magnetic field has finite total power while the corresponding classical formula diverges in the massless limit. We argue that in both cases classical theory describes correctly only the low-frequency part of the spectra, while the total power diverges because of absence of the UV frequency cutoff. Failure of description of gravitational radiation from massless particles by classical General Relativity may be considered as another appeal for quantization of gravity apart from the problem of singularities.

The limit of zero mass of radiating charge in classical electrodynamics is non-trivial. As was discussed recently^{1,2}, the Lienard-Wiechert potentials, due to the factor $(1 - \mathbf{r}\mathbf{v}/r)$ in the denominator, diverge in the direction of the instantaneous velocity for $|\mathbf{v}| = 1$. Attempts to regularize this singularity had not led to reasonable results, so it was argued in² that the truth is that the massless charge does not radiate at all. Similar conjecture was promoted before on different grounds³. This, however, apparently contradicts to infiniteness of the massless limit $\mu \rightarrow 0$ of the well-known formula for synchrotron radiation⁴:

$$P_{cl} = \frac{2e^4 H^2}{3\mu^2} \left(\frac{E}{\mu} \right)^2, \quad (1)$$

where E denotes the energy of the charge, and H – the magnetic field. This discrepancy led us to investigate this problem in the quantum theory⁵.

Radiation from massless charges in quantum electrodynamics is also a non-trivial problem. Within the perturbation theory one encounters, apart from the usual infrared divergencies, the *collinear singularities*, occurring when the photon is emitted from the massless legs of the Feynmann diagrams in the direction of the momentum of a charge⁶. This is manifestation of degeneracy of states of the charge and the photon moving along the

same line. Elimination of collinear divergences is achieved using Kinoshita-Lee-Nauenberg^{7,8} prescription of averaging over an ensemble of degenerate states. Note that the mentioned above line singularities of classical retarded potentials look similar to collinear singularities of quantum theory. Another complication is the screening of the electromagnetic field of the massless charge due to vacuum polarization⁹. However, these problems are evaded, if interaction of the charge with an external electromagnetic field is treated non-perturbatively, using the exact operators in the external classical field. In the case of magnetic field one deals with the bound states of the charge at Landau levels, and the momentum in the plane orthogonal to magnetic field is not conserved. This removes collinear divergences and modifies the propagation function, leading, in particular, to non-zero quantum correction to mass in the one-loop order. In this approach radiation from the massless charge is non-zero and finite⁵.

This analysis also reveals that classical theory still can be applied if only one computes not the Lienard-Wiechert potentials, but their spectral expansion. Indeed, in view of the correspondence principle, it can be expected that classical theory describes correctly the low frequency part of radiation whatever the mass of the radiating charge is. In the massive theory⁴ one has for the low frequencies:

$$\frac{dP}{d\omega} = \frac{e^2 \omega_H 3^{1/6} \Gamma(2/3)}{\pi} \left(\frac{\omega}{\omega_H} \right)^{1/3}, \quad (2)$$

where $\omega_H = eH/E$. This expression depends only on the particle energy and it is unchanged for zero mass. The difference between massive and massless particles, however, is that in the former case the formation length $l \sim 1/(\gamma\omega_H)$ of radiation in a given direction is finite, leading to the frequency cutoff at $\omega_{cr} \sim l^{-1}\gamma^2 \sim \omega_H\gamma^3$, where γ is the Lorentz-factor $\gamma = E/\mu = 1/\sqrt{1-v^2}$. In the massless case $\omega_{cr} \rightarrow \infty$, so there is no frequency cutoff. The Lienard-Wiechert potentials in coordinate representation account for the total field, so they are inappropriate for description of radiation from the massless charge indeed. But performing the spectral decomposition, one can give a reasonable estimate using classical theory with the quantum cutoff $\omega_{quant} = E/\hbar$. Integrating (2) up to this cutoff one obtains:

$$P_{cut} = \int_0^{\omega_{quant}} \frac{dP}{d\omega} d\omega = \frac{e^2 \sqrt{3} \Gamma(2/3)}{4\pi \hbar^2} (3e\hbar H E)^{2/3}. \quad (3)$$

Transition to the massless limit in the quantum theory of synchrotron radiation of massive charge^{4,10} is subtle, since the results of the latter

depend on two dimensionless parameters (in the units $\hbar = c = 1$):

$$f = \frac{H}{H_0} = \frac{eH}{\mu^2}, \quad \chi = \frac{H}{H_0} \frac{E}{\mu} = \frac{eHE}{\mu^3}, \quad (4)$$

diverging as $\mu \rightarrow 0$. Because of this, the standard approximations used in the quasiclassical case (high Landau levels in both initial and final states) in terms of the Macdonald or Airy functions fail, and one has to develop an alternative approximation scheme. We used⁵ the mass-operator in the Schwinger formalism^{10,11} (see also¹²) setting the charge mass to zero *ab initio* and obtaining the spectral power as the integral

$$\frac{dP}{d\omega} = \frac{e^2 v}{4\pi E} \int_0^\infty \left(E^2 (8 - v^2) (1 - v)^2 x \sin \psi + \frac{eHv}{x^2} (1 - \cos \psi) \right) dx, \quad (5)$$

where $v = \omega/E$ and $\psi = \frac{x^3 E^2}{3eH} v(1 - v)^2$. Evaluating the integral over x we get

$$\frac{dP}{d\omega} = \frac{2e^2 \Gamma(2/3)}{27\hbar E} (3e\hbar H E)^{2/3} \mathcal{P}(\hbar\omega/E), \quad (6)$$

where the Planck's constant is restored, and the normalized spectral function is introduced

$$\mathcal{P}(v) = \frac{27}{2\pi\sqrt{3}} v^{1/3} (1 - v)^{2/3}, \quad \int_0^1 \mathcal{P}(v) dv = 1. \quad (7)$$

This spectrum is smooth, exhibiting maximum at

$$\hbar\omega_{\max} = \frac{1}{3} E. \quad (8)$$

The average photon energy is

$$\langle \hbar\omega \rangle = E \int_0^1 v \mathcal{P}(v) dv = \frac{4}{9} E, \quad (9)$$

while the total energy loss per unit time reads

$$P = \int_0^{E/\hbar} P(\omega) d\omega = \frac{2e^2 \Gamma(2/3)}{27\hbar^2} (3e\hbar H E)^{2/3}. \quad (10)$$

It differs from the estimate (3) only by a numerical coefficient. The expression (6) has the following unusual features. It is non-perturbative in the fine structure constant $e^2/\hbar c$, and it has no classical limit $\hbar \rightarrow 0$, being essentially quantum. This could be expected in view of the Eq. (9).

Now we are going to show that gravitational radiation from massless particles exhibits similar features, though, as could be expected from the Weinberg's theorems on soft photons and gravitons⁶, the corresponding

divergencies are weaker (logarithmic). Note that, while consideration of massless charges in electrodynamics appeals rather to the limiting case of the realistic theory, in gravity it is not so, since all particles, including truly massless ones (photons) are subject to gravitational interaction and thus are entitled to emit gravitational radiation. In the textbooks it is tacitly assumed that General Relativity describes correctly gravitational radiation from any classical sources. This turns out not to be true. Namely, the spectrum of gravitational radiation from massless particles moving along null geodesics in curved space-time, being computed classically, is UV divergent. This can be considered as the second argument (apart from the problem of classical singularities) appealing to quantization of gravity.

Gravitational radiation from massive bodies moving along ultrarelativistic geodesics around black holes (gravitational synchrotron radiation, GSR) was considered in early 1970-ies as possible mechanism of enhancement of the flux of gravitational waves from the center of Galaxy to explain the (unconfirmed) Weber's results. Calculations were performed in the Schwarzschild^{13,14} and Kerr^{15,16} geometries. Here we discuss the massless limit in the GSR theory, which is interesting not only in view of the above conceptual problem, but also in view of discovery of high energy astrophysical sources involving strong fluxes of photons and neutrinos.

Consider for simplicity the Schwarzschild case. Timelike geodesics parameterized by the proper time τ obey the radial equation $(dr/d\tau)^2 + U(r) = 0$ with the effective potential

$$U(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + 1\right) - \gamma^2,$$

where $\gamma = E/\mu$ as before and L is the angular momentum. The most interesting are the winding orbits which perform many turns around black hole before being scattered back, or absorbed by the hole. To estimate gravitational radiation one considers circular orbits, whose radii r_p correspond to two conditions $U(r_p) = 0 = U'(r_p)$. Solving these equations with respect to γ and L , one finds $\gamma = (1 - 2M/r_p)(1 - 3M/r_p)^{-1/2}$, $L/\gamma = (Mr_p)^{1/2}(1 - 2M/r)^{-1}$, while the rotation frequency in terms of the orbit radius r_p reads $\omega_0 = d\phi/dt = (M/r_p)^{1/2}$. Relativistic time-like circular orbits with $3M < r_p < 6M$ are unstable. Those which lie in the interval $3M < r_p < 4M$ become unbound under small perturbations, they correspond to large angle scattering with the impact parameter $b = L/(\gamma^2 - 1)^{1/2} = r_p(4M/r_p - 1)^{-1/2}$. In the ultrarelativistic case $\gamma \gg 1$ the unbound orbits with the impact parameter close to the criti-

cal value $b = 3\sqrt{3}M$ scatter on the black hole with multiple revolutions, radiation from these orbits can be estimated using the GSR power. The limit $\gamma = \infty$ corresponds to the massless particles (photon orbits), the corresponding rotation radius being $r_{\text{ph}} = 3M$. Its dislocation with respect to timelike ultrarelativistic orbits can be characterized by a small parameter δ via $r_p = (3 + \delta)M$. Two useful relations then follow in the leading order in δ : $dt/d\tau = \sqrt{3/\delta}$, $\gamma^2 = 1/3\delta$.

Radiation field is expanded in terms of the appropriate angular harmonics, labeled by two integers l, m , $|m| \leq l$, the main contribution for large γ coming from $|m| \gg 1$ and l differing from $|m|$ by 0, 1, depending on polarization. The total intensity can be presented as a sum

$$P_{\text{GSR}} = \sum_{m=1}^{\infty} \frac{E^2 \omega_0}{M} F_m(r_p, M) \frac{m_{\text{cr}}}{m} e^{-m/m_{\text{cr}}}, \quad (11)$$

where each term corresponds to radiation with the frequency $\omega = m\omega_0$, F_m is a smooth function of the parameters, and the critical frequency is

$$m_{\text{cr}} = \frac{12}{\pi} \gamma^2. \quad (12)$$

The spectrum is therefore a falling function of the harmonic number m , and it is cut off classically at the frequency γ times smaller than in the case of flat space synchrotron radiation (because of the increasing formation length due to closeness of ultrarelativistic timelike geodesics of the radiating particle and the null geodesics followed by gravitons¹⁴).

The main contribution (96%) to the total power comes from the polarization commonly denoted as \otimes . This quantity was computed in¹⁶ (in earlier papers only the distribution over the harmonics was given) and reads:

$$P_{\text{GSR}} = \frac{6e^{-\pi/4}(r_p - M) |\Gamma(1/4 + i/4)|^2}{\pi^{3/2} r_p^2 (r_p + 3M)} E^2 \ln(E/\mu). \quad (13)$$

The last factor reflects the logarithmic divergence of the sum (11) if the frequency cutoff (12) is removed, what happens in the limit of zero mass $\gamma = \infty$. We therefore conclude that General Relativity fails to provide full description of gravitational radiation from massless particles. Quantum theory of gravitational synchrotron radiation is not developed yet, but one can hope to get a correct estimate of the total power replacing the divergent logarithm by $\ln(E/\hbar\omega_0)$ as we tested in the electromagnetic case.

Radiation of scalar $s = 0$ and vector $s = 1$ waves is described similarly to Eq. (11) with different m -dependence, namely $(m/m_{\text{cr}})^{1-s}$, with $s = 2$ standing for gravitational waves. Thus the divergence of the total power as

$\mu \rightarrow 0$ is γ^2 for $s = 0, 1$. Softer divergence in the gravitational case is due to the fact that the effective gravitational coupling is proportional to the energy, as was noted by Weinberg long ago.⁶

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